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ON PROJECTIVE MOTION IN A SYMMETRIC AND PROJECTIVE SYMMETRIC FINSLER MANIFOLD

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The present communication has been divided into three sections of which the first section is introductory. The second section has been divoted to the study of "Projective motion in a symmetric finsler manifold". In this section we have derived the conditions which shall be satisfied

(i) When the infinite gimal transformation x -i = x i + ε v i defines a projective motion in a symmetric finsler manifold and

(ii) when the projective motion admitted in asymmetric finsler manifold become an affine motion and in this continuation have also derived the necessory and sufficient conditions which shall be satisfied in order that a symmetric finsler manifold may admit a projective motion. The third section has been devoted to the study of projective motion in a projective symmetric manifold.

In this section we have derived the necessary and sufficient condition under which in a finsler manifold admitting projective motion the Berwald's curvature tensor H i jkh (x, x) becomes a a Lie- invariant and in this continuation have also derived the conditions under which the projective deviation tensor w i z (x , x) becomes a lie-invariant in a finsler manifold admitting projective motion.

In the lost we have absorved that in a projective symmetric manifold if there exists a proper projective motion than the space under consideration must be projectively flat and also that a projective curvature tensor cannot admit a proper projective motion, at the most the manifold under consideration may admit an affine motion.

KEYWORDS: Projective Symmetric Finsler Manifold

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1. INTRODUCTION

The equations of geodesic deviation have been given by Berwald [1] in the form

$$\frac{\delta^2 z^i}{\delta u^2} + H_k^j(x,\dot{x})Z^k = 0, \tag{1.1}$$

where Z^i is called the deviation vector. The tensor $H_k^j(x,\dot{x})$ is called the deviation tensor defined by

$$H_k^j(x,\dot{x}) \stackrel{\text{def}}{=} K_{ihk}^j(x,\dot{x})\dot{x}^i\dot{x}^h. \tag{1.2}$$

It can also be written in the form

$$H_k^i(x,\dot{x}) = 2\partial_k G^i - \partial_k(\dot{\partial}_h G^i)\dot{x}^h + 2G_{km}^i G^m - \dot{\partial}_k G^m \dot{\partial}_m G^i, \tag{1.3}$$

where, we have used the fact that the function $G^m(x,\dot{x})$ is positively homogeneous of degree two in its directional arguments.

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The tensors defined by

$$H^i_{jk}(x,\dot{x}) \stackrel{\text{def}}{=} \frac{1}{3}\left(\dot{\partial}_j H^i_k - \dot{\partial}_k H^i_j\right) \tag{1.4}$$

and $$H^i_{hjk}(x,\dot{x}) \stackrel{\text{def}}{=} \dot{\partial}_h H^i_{jk} \tag{1.5}$$

are explicitly written as

$$H^i_{jk} = \partial_k \dot{\partial}_j G^i - \partial_j \dot{\partial}_k G^i + G^i_{kr}\,\dot{\partial}_j G^r - G^i_{rj}\,\dot{\partial}_k G^r \tag{1.6}$$

and $$H^i_{hjk} = \partial_k G^i_{hj} - \partial_j G^i_{hk} + G^r_{hj} G^i_{rk} - G^r_{hk} G^i_{rj} + G^i_{rhk}\dot{\partial}_j G^r - G^i_{rhj}\dot{\partial}_k G^r,$$

where $G^i_{hjk} = \dot{\partial}_k G^i_{hj}$

and

$$G^i_{hjk}\,\dot{x}^k = 0. \tag{1.7}$$

We can easily obtain the following

(a) $H^i_k(x,\dot{x})\dot{x}^k = 0$, (b) $H^i_{jk}\dot{x}^j = H^i_k$, (c) $H^i_{hjk}\dot{x}^h = H^i_{jk}$,

(d) $H^h_h = H$, (e) $H^h_{ijh} = H_{ij} = \dot{\partial}_i H_j$, (f) $H_i\dot{x}^i = H^i_i = (n-1)H$,

(g) $H^i_{hjk} + H^i_{jkh} + H^i_{khj} = 0$, (h) $H_{jh} - H_{hj} = H^k_{khj}$. **(1.8)**

The commutation formulae involving the tensors H^i_{jkh} and G^i_{jkh} are given by

(a) $T_{(h)(k)} - T_{(k)(h)} = -(\dot{\partial}_i T)H^i_{hk}$,

(b) $T^i_{j(j)(k)} - T^i_{j(k)(h)} = -(\dot{\partial}_r T^i_j)H^r_{hk} + T^r_j H^i_{rhk} - T^i_r H^r_{jhk}$,

(c) $(\dot{\partial}_k T)_{\ell} - \dot{\partial}_k T_{(\ell)} = 0$.

and (d) $(\dot{\partial}_k T^i_j)_{(n)} - \dot{\partial}_k T^i_{j(h)} = T^i_r G^r_{jkh} - T^r_j G^i_{rkh}$. **(1.9)**

2. PROJECTIVE MOTION IN A SYMMETRIC FINSLER SPACE:

The transformation

$$\bar{G}^i(x,\dot{x}) = G^i(x,\dot{x}) - p(x,\dot{x})\dot{x}^i, \tag{2.1}$$

preserves the geodesics of Fn and defines a projective change, $p(x,\dot{x})$ appearing in (2.1) is an arbitrary homogeneous scalar function of degree one in $\dot{x}^i$'s. For $G^i_{jk} = \dot{\partial}_j\dot{\partial}_k G^i$ therefore it can be seen from (2.1) that the Berwald's connection parameters G^i_{jk} under the change (2.1) assume the following from

$$\bar{G}^i_{jk} = G^i_{jk} - 2p_{(j}\delta^i_{k)} - p_{jk}\dot{x}^i, \tag{2.2}$$

where p_k and p_{jk} are the directional derivatives of p and they satisfy

(a) $p_k = \dot{\partial}_k p$, (b) $p_{jk} = \dot{\partial}_j \dot{\partial}_k p$, **(2.3)**

and because of their homogeneous properties these derivatives of the function p satisfy

(a) $p_j \dot{x}^j = p$, (b) $p_{jk} \dot{x}^j = 0$. **(2.4)**

Definition (2.1)

An Fn is called a symmetric manifold if the Berwald's curvature tensor becomes a covariant constants, i.e.

$$H^i_{jkh(m)} = 0. \tag{2.5}$$

It can easily be verified that such a symmetric manifold also satisfies

(a) $H^i_{jk(m)}$, (b) $H^i_{j(m)} = 0$, (c) $H_{(m)} = 0$. **(2.6)**

When the infinitesimal transformation $\bar{x}^i = x^i + \epsilon\, v^i$ admits a projective motion it is seen in [7] that the connection parameters are expressed by

$$\mathcal{L}_v G^i_{jk} = 2\delta^i_{(j} p_{k)} + \dot{x}^i p_{jk}. \tag{2.7}$$

The commutation formula involving the processes of Lie-differentiation and Berwald's covariant differentiation is given by

$$\mathcal{L}_v T^i_{k(j)} - \left(\mathcal{L}_v T^i_k\right)_{(j)} = T^h_k \mathcal{L}_v G^i_{hj} - T^i_h \mathcal{L}_v G^h_{jk} - \left(\dot{\partial}_h T^i_k\right) \mathcal{L}_v G^h_j. \tag{2.8}$$

Applying the commutation formula (2.8) to the deviation tensor H^i_j and thereafter using (2.6) and (2.7), we get

$$\left(\mathcal{L}_v H^i_j\right)_{(k)} = p\dot{\partial}_k H^i_j + p_j H^i_k + 2p_k H^i_j - \delta^i_k p_h H^h_j - \dot{x}^i p_{jh} H^h_k. \tag{2.9}$$

Here, we have taken into account the facts given by (1.8) and (2.4).

Contracting (2.9) with respect to the indices i and j and then using (2.3) and (2.4), we get

$$p\dot{\partial}_k H + 2p_k H - (\mathcal{L}_v H)_{(k)} = 0. \tag{2.10}$$

Transvecting (2.10) by $\dot{x}^k$ and thereafter using (1.9) and (2.4a), we have

Theorem (2.1)

If the infinitesimal transformation $\bar{x}^i = x^i + \epsilon\, v^i$ defines a projective motion in a symmetric Finsler manifold then there exists a scalar function p satisfying

$$p = \dot{x}^k (\mathcal{L}_v H)_{(k)} / 4H, \tag{2.11}$$

for a non-vanishing scalar function H.

It can easily be verified from (2.7) that the vanishing of the vector p_j is the necessary and sufficient condition in order that the projective motion may become an affine motion. Hence, with the help of (2.11) and (2.3a), we can state:

Theorem (2.2)

In order that a projective motion admitted in a symmetric Finsler manifold becomes an affine motion in the same space, it is necessary and sufficient that $\dot{x}^k(\mathcal{L}_v H)_{(k)}$ must vanish.

The expression $\dot{x}^k(\mathcal{L}_v H)_{(k)}$ can be expanded with the help of (2.8) in the following form

$$\dot{x}^k(\mathcal{L}_v H)_{(k)} = \dot{x}^k\{(\dot{\partial}_h H)v^h_{(m)}\dot{x}^m\}, \tag{2.12}$$

where, we have taken into account (2.6c).

Using (1.9) and (2.6c) into the right hand side of (2.12), we get

$$\dot{x}^k(\mathcal{L}_v H)_{(k)} = (\dot{\partial}_h H)(v^h_{(m)(k)})\dot{x}^k\dot{x}^m. \tag{2.13}$$

Using (2.13) in (2.11), we get

$$p = (\dot{\partial}_h H)(v^h_{(m)(k)})/4H. \tag{2.14}$$

With the help of (2.14), we can state:

Theorem (2.3)

In a symmetric Finsler space a projective motion reduces into an affine motion provided the vector field v^i be supposed to be a covariant constant.

Theorem (2.1) stated above can be proved entirely in a different manner.

Taking the covariant derivative of $\mathcal{L}_v H^i_{jk}$ and thereafter using (1.9), we get

$$(\mathcal{L}_v H^i_{jk})_{(m)} = 2H^i_{s[k}v^s_{((m))j]} - H^s_{jk}v^i_{(m)(s)} + H^i_{jkh}v^h_{(m)(s)}\dot{x}^s, \tag{2.15}$$

where, we have taken into account (2.5) and (2.6a).

Using (2.3) and (2.7) in (2.15) and thereafter rearranging the terms, we get

$$\begin{aligned}(\mathcal{L}_v H^i_{jk})_{(m)} = {} & 2p_{[k}H^i_{j]m} - \delta^i_m p_s H^s_{jk} + 2p_m H^i_{jk} + pH^i_{jkm} + 2p_{m[k}H^i_{j]s}\dot{x}^s - \\ & -\dot{x}^i p_{ms}H^s_{jk} + v^h\{H^s_{jk}H^i_{hsm} - 2H^s_{hm[k}H^i_{j]s} - H^i_{jks}H^s_{hm}\} + \\ & +v^h_{(r)}\{H^s_{jk}G^i_{msh} - 2H^i_{s[k}G^s_{j]mh}\}\dot{x}^r.\end{aligned} \tag{2.16}$$

Taking into account the commutation formula given by (1.9) for H^i_{jk}, we get

$$2H^i_{j[(m)(h)]} = H^s_{jk}H^i_{hms} - 2H^s_{hm[k}H^i_{j]s} - H^i_{jks}H^s_{hm} \tag{2.17}$$

And $(\dot{\partial}_h H^i_{jk})_{(m)} - \dot{\partial}_h H^i_{jk(m)} = H^s_{jk}G^i_{hms} - 2H^i_{s[k}G^s_{j]hm}$, **(2.18)**

respectively.

Using (2.5) and (2.6a) in (2.17) and (2.18), we respectively get

$$H^s_{jk}H^i_{hms} - 2H^s_{hm[k}H^i_{j]s} - H^i_{jks}H^s_{hm} = 0, \quad \textbf{(2.19)}$$

and $H^s_{jk}G^i_{hms} - 2H^i_{s[k}G^s_{j]hm} = 0.$ **(2.20)**

Using (2.19) and (2.20) in (2.16), we get

$$(\mathcal{L}_v H^i_{jk})_{(m)} = 2p_{[k}H^i_{j]m} - \delta^i_m p_s H^s_{jk} + 2p_m H^i_{jk} + pH^i_{jkm} +$$

$$+2p_{m[k}H^i_{j]s}\dot{x}^s - \dot{x}^i p_{ms}H^s_{jk}. \quad \textbf{(2.21)}$$

Transvecting (2.21) by $\dot{x}^k$ and then using (2.4b), (2.9) and the fact that $H^i_{jk}\dot{x}^j\dot{x}^k = 0$, (2.21) reduces into (2.9) and thus we shall get Theorem (2.1).

Writing the Lie-derivative of the tensor field H^i_{jk} and thereafter transvecting it by $\dot{x}^h$, we get

$$\mathcal{L}_v H^i_{jk} = 2\dot{x}^h(\mathcal{L}_v G^i_{[k(h)})_{j]}. \quad \textbf{(2.22)}$$

Using (2.4b) and (2.7) in (2.22), we get

$$\mathcal{L}_v H^i_{jk} = 2\{\delta^i_{[k}p_{(j)]} + \dot{x}^i p_{[k(j)]}\}. \quad \textbf{(2.23)}$$

Transvecting (2.23) successively by $\dot{x}^j\dot{x}^k$, we get

$$\mathcal{L}_v H = -\dot{x}^i p_{(i)}. \quad \textbf{(2.24)}$$

Using (2.11) in the covariant derivative of (2.24), we can therefore state:

Theorem (2.4)

The infinitesimal transformation $\bar{x}^i = x^i + \epsilon v^i$ defining a projective motion in a symmetric Finsler manifold satisfies

$$4pH + \dot{x}^i\dot{x}^m p_{(i)(m)} = 0. \quad \textbf{(2.25)}$$

Noting (1.9), (2.4a), (2.23) and the partial derivative of (2.7) to the partial derivative of (2.7) to the Lie-derivative of (2.20), we get

$$2\delta^i_{[j}G^h_{k]sm}p_{(h)} + 2\dot{x}^i(G^i_{smj}p_{[h(k)]} + G^h_{smk}p_{[j(h)]}) + 2(\delta^i_{(s}p_{m)h} + \dot{x}^i p_{hsm})H^h_{jk} -$$

$$-2p_{j(m}H^i_{s)k} + 2p_{k(m}H^i_{s)j} - p_{sm}H^i_{jk} + 2p_{sm[j}H^i_{k]} = 0. \quad \textbf{(2.26)}$$

Contracting (2.26) with respect to the indices i and s and then using (2.4a), we get the following form of (2.26)

$$(n-1)p_{im}H^i_{jk} + 2p_{m[j}H_{k]i} + 2p_{i[j}H^i_{k]m} + 2p_{im[j}H^i_{k]} = 0. \quad \textbf{(2.27)}$$

Transvecting (2.27) by $\dot{x}^k$ and thereafter using (1.8) and (2.4b), we get

$$np_{im}H^i_j - p_{ij}H^i_m = (n-1)p_{mj}H. \quad \textbf{(2.28)}$$

Interchanging the indices m and j in (2.28), we get

$$n p_{ij} H^i_m - p_{im} H^i_j = (n-1) p_{jm} H. \tag{2.29}$$

Since the tensor p_{mj} is symmetric in its lower indices, the right hand side of (2.28) and (2.29) are therefore same. On equating (2.28) and (2.29), we therefore get

$$p_{im} H^i_j = p_{ij} H^i_m. \tag{2.30}$$

Using (2.28) and (2.30), we get

$$p_{im} H^i_j = p_{mj} H. \tag{2.31}$$

We can therefore state:

THEOREM (2.5)

In order that a symmetric Finsler manifold may admit a projective motion, it is necessary and sufficient that (2.30) and (2.31) should hold together.

3. PROJECTIVE MOTION IN A PROJECTIVE SYMMETRIC FINSLER MANIFOLD

Berwald's curvature tensor H^i_{jkh} being a tensor of the type (1.3), its Lie-derivative can be obtained from the formula given by (2.8). On the other hand the Lie-derivative of H^i_{jkh} is also expressed according to Yano [8] by

$$2\left(\mathcal{L}_v G^i_{[kh}\right)_{(j)]} = \mathcal{L}_v H^i_{jkh} + 2\left(\mathcal{L}_v G^s_{m\,[j}\right)\dot{x}^m G^i_{k]sh}. \tag{3.1}$$

Transvecting (3.1) by $\dot{x}^h$ and thereafter using (1.9) and the relation $\mathcal{L}_v G^i_j = \delta^i_j p + \dot{x}^i p_j$, we get

$$\mathcal{L}_v H^i_{jk} = 2\{\delta^i_{[k} p_{(j)]} + \dot{x}^i p_{[k(j)]}\}, \tag{3.2}$$

in a projective Finsler manifold.

Using (1.9) in (3.2), we get

$$\mathcal{L}_v H^i_j = 2\dot{x}^i p_{(j)} - \delta^i_j \dot{x}^k p_{(k)} - \dot{x}^i \dot{x}^k p_{j\,(k)}, \tag{3.3}$$

and $\mathcal{L}_v H = -\dot{x}^i p_{(j)}$, (3.4)

respectively. Therefore, we can state:

THEOREM (3.1)

In an Fn admitting a projective motion, in order that H^i_{jkh} may become a Lie-invariant it is necessary and sufficient that $p_{k(j)} = 0$.

We now proceed to prove the above statement. Finsler manifold Fn admitting a projective motion along with $p_{k(j)} = 0$ reduces (3.2) to $\mathcal{L}_v H^i_{jk} = 0$, which after making use of (1.9) gives $\mathcal{L}_v H^i_{jkh} = 0$.

In order to prove the sufficient part of this theorem, we have $\mathcal{L}_v H^i_{jkh} = 0$ in a projective Finsler manifold. Transvecting this relation by $\dot{x}^h, \dot{x}^k$ successively we get $\mathcal{L}_v H^i_j = 0$ where we have taken into consideration (1.9), thereafter

contracting this relation with respect to the indices i and j and thereafter using (1.9), we get $\mathcal{L}_v H = 0$.

In view of this fact, from (3.4) we get $\dot{x}^i p_i = 0$. The partial derivative of this relation with respect to $\dot{x}^j$ after making use of (1.9) and (2.3a) gives

$$p_{(j)} + \dot{x}^i p_{j(i)} = 0. \quad \textbf{(3.5)}$$

Contracting (3.2) with respect to the indices i and k and then using $\mathcal{L}H_j = 0$, we get

$$np_{(j)} - \dot{x}^i p_{j(i)} = 0, \quad \textbf{(3.6)}$$

where we have taken into account (2.3a) and the invariant property of $\dot{x}^i$under the process of Berwald covariant differentiation using (3.5) and (3.6), we get $p_{(j)} = 0$, from which we can easily get the required result.

The tensor defined by

$$W_k^j \overset{\text{def}}{=} H_k^j - H\delta_k^j - \frac{1}{n+1}(\dot{\partial}_i H_k^i - \dot{\partial}_k H)\dot{x}^j, \quad \textbf{(3.7)}$$

is called the projective deviation tensor. Forming its Lie-derivative and using the relations (3.3), (3.4), (2.4), we can state:

THEOREM (3.2):

In a Finsler space admitting a projective motion the projective derivation tensor W_j^i becomes a Lie-invariant, i.e.

$$\mathcal{L}_v W_j^i = 0. \quad \textbf{(3.8)}$$

DEFINITION (3.1)

A Finsler manifold characterised by

$$W_{jkh(s)}^i = 0, \quad \textbf{(3.9)}$$

has been called by Mishra [6] as projectively symmetric manifold and such a manifold is denoted by $PS - F_n$.

It has been seen that a projective symmetric manifold also admits

$$\text{(a) } W_{jk(s)}^i = 0, \qquad \text{(b) } W_{j(s)}^i = 0. \quad \textbf{(3.10)}$$

The commutation formula involving the processes of Lie-differentiation and Berwald's covariant differentiation for an arbitrary contravariant X^i is given as

$$\mathcal{L}_v(X_{(k)}^i) - (\mathcal{L}_v X^i)_{(k)} = X^r \mathcal{L}_v G_{rk}^i - (\dot{\partial}_r X^i)\mathcal{L}_v G_{sk}^r \dot{x}^s. \quad \textbf{(3.11)}$$

Applying (3.11) for W_j^i, we get

$$\mathcal{L}_v W_{j(k)}^i - (\mathcal{L}_v W_j^i)_{(k)} = W_j^s \mathcal{L}_v G_{sk}^i - W_s^i \mathcal{L}_v G_{jk}^s - (\dot{\partial}_s W_j^i)\mathcal{L}_v G_{mk}^s \dot{x}^m. \quad \textbf{(3.12)}$$

Now, we take into account the existence of a projective motion in a projective symmetric manifold, in such manifold (2.7), (3.7) and (3.10a) always holds. Making use of these relations in (3.12), we get

$$W_j^s\{2\delta^i_{(s}p_{k)} + \dot{x}^i p_{sk}\} - W_s^i\{2\delta^s_{(j}p_{k)} + \dot{x}^s p_{jk}\} -$$
$$-(\dot{\partial}_s W_j^i)\{2\delta^s_{(k}p_{m)} + \dot{x}^s p_{km}\}\dot{x}^m = 0. \quad \textbf{(3.13)}$$

It has been observed that the projective deviation tensors W_j^i satisfies the following relations

$$\text{(a) } W_i^i = 0, \qquad \text{(b) } W_k^j \dot{x}^k = 0, \quad \text{(c) } \dot{\partial}_i W_k^i = 0. \quad \textbf{(3.14)}$$

Using (3.14) and (2.3a) in (3.13), we get

$$\delta^i_k p_s W_j^s + \dot{x}^i p_{sk} W_j^s - p_j W_k^i - 2p_k W_j^i - p\dot{\partial}_k W_j^i = 0. \quad \textbf{(3.15)}$$

Contracting (3.15) with respect to the indices i and k and thereafter using (2.4b) and the fact that $\dot{\partial}_s W_j^i \dot{x}^s = 2W_j^i$, we get

$$p_s W_j^s = 0. \quad \textbf{(3.16)}$$

Transvecting (3.15) by $\dot{x}^k$ and then using (3.14) and (2.4b), we get

$$\dot{x}^i p_s W_j^s = 4pW_j^i. \quad \textbf{(3.17)}$$

Using (3.16) in (3.17), we get

$$pW_j^i = 0. \quad \textbf{(3.18)}$$

An obvious consequence of (3.18) is either $p = 0$ or $W_j^i = 0$. But $p = 0$ reduces a projective motion into an affine motion while the vanishing of the tensor W_j^i tells that the space under consideration is projectively flat. Therefore, we can state:

THEOREM (3.3)

In a projective symmetric Finsler space if there exists a proper projective motion i.e. those with $p \neq 0$ then the space under consideration must be projectively flat.

THEOREM (3.4)

A projective symmetric manifold with non-zero projective curvature cannot admit a proper projective motion, at the most the manifold under consideration may admit an affine motion.

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